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# Covariant And Local Field Theory On The World Sheet <sup>1</sup>

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## Abstract

In earlier work, using the light cone picture, a world sheet field theory that sums planar  $\phi^3$  graphs was constructed and developed. Since this theory is both non-local and not explicitly Lorentz invariant, it is desirable to have a covariant and local alternative. In this paper, we construct such a covariant and local world sheet field theory, and show that it is equivalent to the original non-covariant version.

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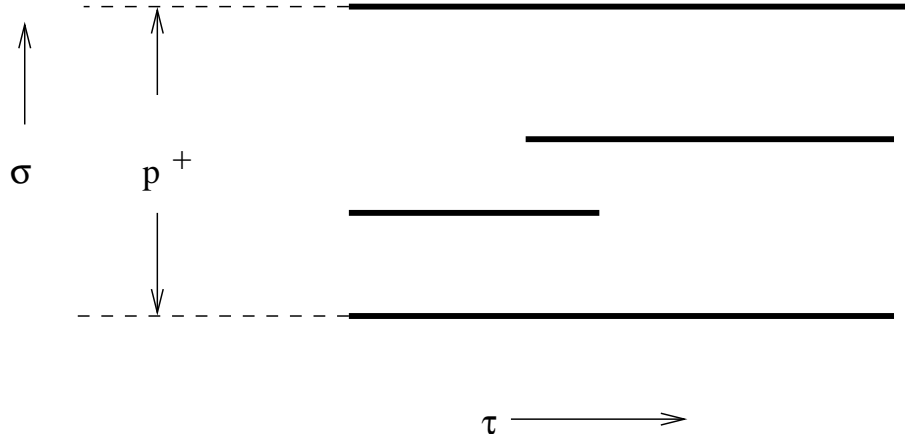


Figure 1: A Typical Graph

## 1. Introduction

In this article, we continue the project of putting planar field theory on the world sheet. The new feature of the present work is that for the first time, an explicitly Lorentz invariant treatment is presented. This is an important advance, since, most of the earlier work, following t'Hooft's seminal paper [1], dealt with the planar  $\phi^3$  field theory in the light cone frame. Although it had many convenient features, this approach had the disadvantage of not being explicitly Lorentz covariant. A manifestly covariant treatment was sketched in [2], but this was done in the framework of the first quantized approach. As explained in section 2, on the world sheet, a typical graph of the planar  $\phi^3$  theory can be pictured as a collection of parallel solid lines, which forms the boundaries of the propagators (Fig.1). The momenta carried by the graph run through these lines, and in the first quantized approach, these are treated as fields and quantized. There is, however, a drawback to this approach: The interaction takes place by the creation and annihilation of the solid lines (boundaries), and this is difficult to accommodate in a first quantized theory. In reference [3], this problem was overcome by developing a second quantized theory, which naturally incorporates the creation and destruction of the boundaries. One is able to make quite a bit of progress with this approach; for example, in [4], using the mean field approximation, this model was shown to lead to string formation.

The second quantized theory described above was constructed in the light

cone framework, and although it is intrinsically Lorentz invariant, it has again the disadvantage of not being manifestly covariant. Any approximation scheme, such as the mean field method, is likely to spoil Lorentz invariance. The model is also non-local on the world sheet, which complicates matters. Also, there is a spurious singularity at  $p^+ = 0$  that is typical of the field theories in the light cone coordinates. This singularity drops out of the physical quantities in the exact theory, but it is problematic in any approximation scheme. Finally, the theory has to be renormalized in higher dimensions, and renormalization a non-covariant theory is notoriously difficult<sup>3</sup>.

The approach we will develop does not suffer from any of the difficulties listed above. It is a second quantized theory that is explicitly covariant and local on the world sheet. Needless to say, it does not have spurious singularities. It is therefore well suited to an approximation scheme such as the mean field method. To keep the paper to a reasonable length, we will not attempt to carry out such a calculation here, and leave it as an interesting project for future research.

A good portion of the paper is devoted to the review of the model in the light cone picture. This is necessary to make the paper self contained, but also, in reviewing the light cone picture, we will develop the tools needed for the covariant approach. Finally, we prove the validity of the covariant approach by showing that it is equivalent to the well established light cone model.

In section 2, we briefly review the world sheet picture of the graphs of the planar  $\phi^3$  theory in the light cone variables. Section 3 is a review of the first quantized world sheet formalism developed in [2], which reproduces the free massless propagators of the light cone picture. In particular, we emphasize the analogy with the one dimensional electrostatics. In this analogy, the momenta correspond to the electrostatic potential, which is generated by charges residing on the boundaries. These charges will play an important role in the next section, where we construct a second quantized world sheet field theory. As we have already emphasized, the interaction creates and destroys solid lines (boundaries), and therefore second quantization becomes indispensable. We follow the ideas introduced in [3] and define fields that create and destroy boundaries. A crucial difference is that, in contrast to [3], these fields are labeled by the charges, and not by the potentials. This enables us to write down a simple local expression for the free massless action

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<sup>3</sup>See [5] for a regulator that preserves Lorentz invariance.

(eq.(9)).

In section 5, we introduce the boundary changing interaction (eq.(12)). However, this interaction is not quite correct, since it generates multiple solid lines which lead to over counting. This problem was overcome in [7] by introducing world sheet fermions. The interaction is then modified by fermionic terms (eq.(17)), and Fermi statistics forbids the unallowed states. The constraint that projects out the unallowed states is expressed by eq.(21).

There is, however, another problem with the interaction that is not so easily cured: The prefactor  $1/(2p^+)$  in eq.(1), which has to be attached to the interaction, is missing. To try to incorporate this factor would lead to complicated non-local terms. Instead, we observe that this factor is needed for Lorentz invariance in the light cone picture. In the covariant model we will develop, there is no such extra factor. Since we regard the present non-covariant model only as a laboratory in preparation for the eventual covariant theory, we will leave its present simple though imperfect form.

So far, we have been dealing with a massless model. In section 6, we add a simple local mass term to the action. This is done by introducing an extra spacelike dimension  $s$ , and coupling it to electric dipoles located on the boundaries. The dipole moment of these dipoles is proportional to the mass. This completes the electrostatic analogy: The charges on the boundaries generate the momentum dependent term in the propagator, and the dipoles generate the mass term. However, the equations of motion for  $s$  (eq.(26)), in addition to the desired mass term, allow for unwanted solutions. We show how these extra solutions can be projected out by means of an orbifold projection (eq.(27)).

Starting with section 7, we will develop the covariant world sheet theory that has been our goal. We start with the first quantized picture of the world sheet. The first step is to cast the free massless light cone action of eq.(2) into a covariant form. This was already done in [2], with the result given by eq.(28). In section 7, we present a review of the proof of equivalence of the covariant and the light cone actions given in [2]. The proof consists of showing that the additional field  $\lambda$  that was introduced in the covariant version can be set equal to one. At the same time, the coordinates  $\sigma$  and  $\tau$  can be fixed at their light cone values  $p^+$  and  $x^+$ , completing the proof of equivalence. Invariance of the action under restricted  $\sigma$  and  $\tau$  reparametrizations play an important role in reaching this result. For the second quantized covariant version of the model, we only need to fix  $\lambda = 1$ ; the coordinates  $\sigma$  and  $\tau$  are left arbitrary. Finally, we also give the covariant version of the mass term.

In section 8, we show how to convert the first quantized covariant theory (eqs.(52) and (53)) into a second quantized one. This amounts to almost repeating the derivation given for the light cone version in sections 4 and 5. Essentially, all one has to do is to replace the transverse momenta  $\mathbf{q}$  and the charge  $\mathbf{z}$  by the covariant vectors  $q^\mu$  and  $z^\mu$ . This leads to the main result of this paper: A covariant second quantized theory, with the action (62), supplemented by constraints on the states (eq.(61)). Finally, we argue that the constraint resulting from varying the action with respect to  $\lambda$  should be reintroduced in a suitably weakened form. Section 9 summarizes our conclusions and possible future directions of research.

## 2. The Light Cone World Sheet Picture

We start with a brief review of the representation of the planar graphs of the  $\phi^3$  theory in the light cone variables on the world sheet [1]. Although in this paper we are mainly interested in covariant Feynman graphs, we do not know how to cover the world sheet of the interacting theory with these graphs directly in a manifestly covariant picture. Instead, we choose an indirect route: We first ansatz a covariant world sheet theory, and then by suitably fixing coordinates, show that it is equivalent to the well established light cone picture. In this picture, the world sheet is parametrized by the light cone coordinates  $\tau = x^+$  and  $\sigma = p^+$  as a collection of solid lines (Fig.1), where the  $n$ 'th line carries a  $D - 2$  dimensional transverse momentum  $\mathbf{q}_n$ . Two adjacent solid lines labeled by  $n$  and  $n+1$  correspond to the light cone propagator

$$\Delta(\mathbf{p}_n) = \frac{\theta(\tau)}{2p_n^+} \exp\left(-i\tau \frac{\mathbf{p}_n^2 + m^2}{2p_n^+}\right), \quad (1)$$

where  $\mathbf{p}_n = \mathbf{q}_n - \mathbf{q}_{n+1}$  is the transverse momentum flowing through the propagator, and the width of the strip is  $p_n^+$ , the  $+$  component of the momentum carried by the propagator. A factor of the coupling constant  $g$  is inserted at the beginning and at the end of each line, where the interaction takes place. Ultimately, one has to integrate over all possible locations and lengths of the solid lines, as well as over the momenta they carry. We note that the cubic interaction vertex is the same as the corresponding open string vertex in the light cone picture.

There is a spurious singularity peculiar to the light cone picture at  $p^+ = 0$ . One way to avoid this singularity is to discretize the coordinate  $\sigma$  in steps of length  $a$ . A useful way of visualizing the discretized world sheet is pictured

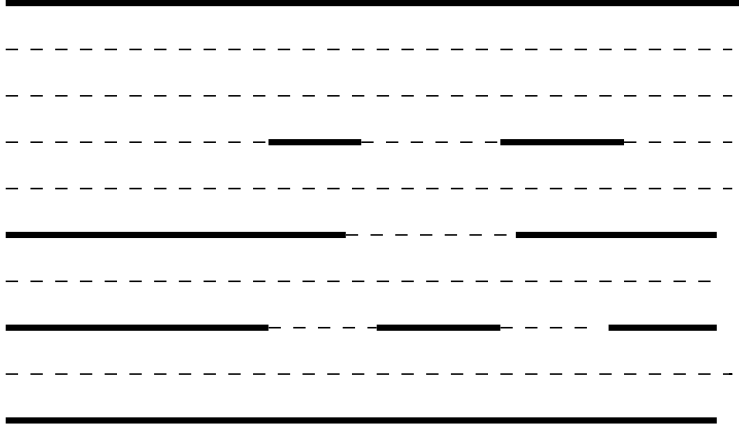


Figure 2: Solid And Dotted Lines

in Fig.2. The boundaries of the propagators are marked by solid lines, and the bulk is filled by dotted lines spaced at a distance  $a$ . In the covariant picture, there is of course no singularity at  $p^+ = 0$ , and we will mostly take  $\sigma$  to be continuous, although we still occasionally use discretization as a convenient way to describe the world sheet. In this article, the world sheet field theory will be treated classically, and so the formal passage from discrete to continuous  $\sigma$  will present no problem. Of course, if higher order corrections to the classical picture are considered, there will in general be divergences which need to be renormalized. One has then to introduce some kind of regulator, possibly the discretized world sheet or something else to study the continuum limit more carefully. We hope to address this important problem in the future.

### 3. Light Cone World Sheet Field Theory For The Free Massless Scalar

In this and the next section, we will develop a world sheet field theory which will reproduce the light cone graphs described earlier, with the exception of the prefactor  $1/(2p^+)$ . This will serve as a stepping stone for the Lorentz covariant theory which is the focus of our interest. We will see that having understood this simplified model, it will be easy to generalize it to a fully covariant theory. We first briefly review the world sheet formalism developed in [2]. Consider first in the light cone variables the non-interacting massless model, which is a collection of infinitely extended propagators. The

corresponding world sheet action is given by

$$S_0 = \int d\tau \int d\sigma \left( -\frac{1}{2}(\partial_\sigma \mathbf{q})^2 + \sum_n \delta(\sigma - \sigma_n) \mathbf{y}_n \cdot \partial_\tau \mathbf{q} \right). \quad (2)$$

Here  $\sigma_n$  mark the boundaries of these propagators (solid lines), and  $\mathbf{y}_n$  are Lagrange multipliers ensuring that the momentum  $\mathbf{q}(\sigma_n, \tau) = \mathbf{q}_n(\tau)$ , flowing through the  $n$ 'th solid line is conserved in time. Solving the classical equation of motion

$$\partial_\sigma^2 \mathbf{q} = \sum_n \delta(\sigma - \sigma_n) \partial_\tau \mathbf{y}_n \quad (3)$$

for  $\mathbf{q}(\sigma, \tau)$  in the interval  $\sigma_n \leq \sigma \leq \sigma_{n+1}$  in terms of its  $\tau$  independent boundary values  $\mathbf{q}_n = \mathbf{q}(\sigma_n)$  leads to the result

$$\mathbf{q}(\sigma, \tau) = \frac{(\sigma - \sigma_n) \mathbf{q}_{n+1} + (\sigma_{n+1} - \sigma) \mathbf{q}_n}{\sigma_{n+1} - \sigma_n}. \quad (4)$$

Substituting this in the action gives

$$S_0 \rightarrow -\frac{1}{2}(\tau_f - \tau_i) \sum_n \frac{(\mathbf{q}_{n+1} - \mathbf{q}_n)^2}{\sigma_{n+1} - \sigma_n}. \quad (5)$$

Identifying

$$\sigma_{n+1} - \sigma_n = p_n^+, \quad \mathbf{q}_{n+1} - \mathbf{q}_n = \mathbf{p}_n,$$

and  $\tau_{i,f}$  with the initial(final) time, we recover eq.(1).

These equations admit a useful one dimensional electrostatic analogy. The function  $\mathbf{q}$  can be identified with the electrostatic potential, and after integrating by parts with respect to  $\tau$  in eq.(2),

$$\mathbf{z}_n = -\partial_\tau \mathbf{y}_n \quad (6)$$

can be identified with line charges located at on the boundaries at  $\sigma = \sigma_n$ . We note that these charges are conserved:

$$\partial_\tau \mathbf{z}_n = 0.$$

If so desired, both  $\mathbf{q}$  and the action can be expressed in terms of the charges. For example, from eq.(3)

$$\mathbf{q}(\sigma, \tau) = \frac{1}{2} \sum_n |\sigma - \sigma_n| \mathbf{z}_n(\tau). \quad (7)$$

At this point, one is free to choose either the potentials  $\mathbf{q}(\sigma_n, \tau)$  or the charges  $\mathbf{z}_n$  as independent dynamical variables. Using the above set of equations, it is easy to transform from one set of variables to the other set.

#### 4. Second Quantization

The problem with the formalism described above is that it is difficult to introduce the interaction. We recall that the interaction takes place where a solid line begins or ends, and the picture developed so far cannot accommodate creation or destruction of the solid lines (boundaries). For this purpose, we need a second quantized theory with fields that create and destroy the boundaries. Such a second quantized theory was introduced in [3] and developed further in [4]. Here, we will make use of the same idea, with however, an important difference. In [3], the fields were labeled, in addition to  $\sigma$  and  $\tau$ , by the momenta (potentials)  $\mathbf{q}$ . Here, instead of  $\mathbf{q}$ , we will label the fields by the charge  $\mathbf{z}$ . Accordingly, we introduce a complex scalar field and its conjugate, which satisfy the commutation relations

$$[\phi(\sigma, \tau, \mathbf{z}), \phi^\dagger(\sigma', \tau, \mathbf{z}')] = \delta(\sigma - \sigma') \delta(\mathbf{z} - \mathbf{z}'). \quad (8)$$

The field  $\phi$  destroys a solid line at  $\sigma$  carrying a charge  $\mathbf{z}$  and  $\phi^\dagger$  creates such a line. Vacuum is the empty world sheet (all dotted lines), annihilated by the  $\phi$ 's.

The motivation for switching from  $\mathbf{q}$  to  $\mathbf{z}$  in labeling fields is to try to simplify the resulting field theory. In reference [3], the world sheet field theory in the  $\mathbf{q}$  basis contained complicated non-local terms. We have found it difficult to convert these terms into a manifestly covariant form. In contrast, in  $\mathbf{z}$  basis, a simple local theory emerges, and this can easily be generalized to a covariant form. We start with the second quantized version of eq.(2):

$$\begin{aligned} S_0 &= \int d\tau \left( i \int d\sigma \int d\mathbf{z} \phi^\dagger \partial_\tau \phi - H_0(\tau) \right), \\ H_0 &= \int d\sigma \left( \frac{1}{2} (\partial_\sigma \mathbf{q})^2 + \mathbf{q} \cdot \int d\mathbf{z} \phi^\dagger \mathbf{z} \phi \right). \end{aligned} \quad (9)$$

The  $\mathbf{z}$  integration is over the  $D - 2$  dimensional transverse space.

To see that this is the right action, we first note that the first term in the equation for  $S_0$  implies the canonical commutation relations between  $\phi$  and  $\phi^\dagger$  (eq.(8)). Next, consider the state  $|s\rangle$  at some fixed  $\tau$ , with a collection of



solid lines placed at points  $\sigma = \sigma_n$ :

$$|s\rangle = \prod_n \phi^\dagger(\sigma_n, \mathbf{z}_n) |0\rangle. \quad (10)$$

It is easy to show that

$$\mathbf{q} \cdot \int d\mathbf{z} \phi^\dagger \mathbf{z} \phi |s\rangle = \sum_n \delta(\sigma - \sigma_n) \mathbf{q} \cdot \mathbf{z}_n |s\rangle, \quad (11)$$

and remembering that  $\mathbf{z}_n = -\partial_\tau \mathbf{y}_n$ ,  $H_0$ , acting on  $|s\rangle$ , exactly reproduces the action of eq.(2).

There is an important restriction on the state  $|s\rangle$ : Each site  $\sigma_n$  is singly occupied by the corresponding  $\phi^\dagger(\sigma_n, \mathbf{z}_n)$ . This is easier to see if  $\sigma$  is discretized; multiple solid lines at the same site would lead to overcounting. We will call such a state an allowed state. In a free model, if we require all the states at some initial time to be allowed, their time development will still leave them as allowed states. We will see that things are not so simple in the presence of interaction.

## 5. The Interaction

The simplest way to introduce interaction is to set

$$H_I = \int d\sigma \int d\mathbf{z} g \left( \phi(\sigma, \mathbf{z}) + \phi^\dagger(\sigma, \mathbf{z}) \right), \quad (12)$$

where  $g$  is the coupling constant. To show that this reproduces the the perturbation expansion, we expand in powers of  $g$ , using the interaction representation. In this representation,  $\phi$  is a free field that satisfies the equation

$$\partial_\tau \phi = 0,$$

and the commutation relations (8). Acting on the states (10),  $\phi^\dagger$  creates a solid line and  $\phi$  annihilates one and, expanding in powers of  $g$  in the interaction representation generates the perturbation graphs pictured in Fig.2.

However, this interaction would also create unallowed states with multiple solid lines. One way to overcome this problem is to modify the interaction by adding fermionic fields [7]. We introduce two fermionic fields and their conjugates, which satisfy the anti commutation relations

$$[\psi_i(\sigma, \tau), \psi_{i'}^\dagger(\sigma', \tau)]_+ = \delta_{i,i'} \delta(\sigma - \sigma'), \quad (13)$$

with  $i, i' = 1, 2$ . The structure of the fermionic states is best visualized when  $\sigma$  is discrete: The site of a dotted line is occupied by one  $\psi_1^\dagger$  and the site of a solid line by one  $\psi_2^\dagger$ . Thus, the state  $|0\rangle$  corresponding to the empty world sheet (all dotted lines) satisfies

$$\psi_2|0\rangle = 0, \quad \psi_1^\dagger|0\rangle = 0, \quad \phi|0\rangle = 0, \quad (14)$$

for all  $\phi$ 's and  $\psi$ 's. We now modify eq.(10) to redefine an allowed state at a fixed  $\tau$ :

$$|s_a\rangle = \prod_n \left( \phi^\dagger(\sigma_n, \mathbf{z}_n) \psi_2^\dagger(\sigma_n) \right) |0\rangle. \quad (15)$$

Multiply occupied lines are now forbidden by Fermi statistics.

To ensure that allowed states develop into other allowed states, one has to modify the interaction (eq.(12)). We define the composite operators

$$\rho^+ = \psi_1^\dagger \psi_2, \quad \rho^- = \psi_2^\dagger \psi_1, \quad \rho = \psi_2^\dagger \psi_2, \quad (16)$$

and with their help, we rewrite the interaction as

$$H_I = \int d\sigma \int d\mathbf{z} g \left( \rho^+(\sigma) \phi(\sigma, \mathbf{z}) + \rho^-(\sigma) \phi^\dagger(\sigma, \mathbf{z}) \right). \quad (17)$$

This new interaction hamiltonian, as well as the total hamiltonian

$$H = H_0 + H_I \quad (18)$$

maps allowed states into other allowed states. One way to see this is to notice that the operator

$$K(\sigma, \tau) = \int d\mathbf{z} \phi^\dagger(\sigma, \tau, \mathbf{z}) \phi(\sigma, \tau, \mathbf{z}) - \rho(\sigma, \tau) \quad (19)$$

commutes with  $H$ ,

$$[K, H] = 0 \quad (20)$$

and annihilates the allowed states:

$$K|s_a\rangle = 0. \quad (21)$$

Here, we have made use of

$$\rho(\sigma)|s_a\rangle = \left( \sum_n \delta(\sigma - \sigma_n) \right) |s_a\rangle. \quad (22)$$

Eq.(21), imposed at a fixed  $\tau$ , can serve as the definition of the allowed states at that  $\tau$ . Since  $K$  commutes with  $H$ , this condition then holds for all  $\tau$ .

In [4], we imposed a condition analogous to (21), with  $\mathbf{z}$  replaced by  $\mathbf{q}$ . Actually, we had a stronger version of it

$$K = 0,$$

which was implemented by multiplying with a Lagrange multiplier and adding it to the action. This has certain computational advantages, but here we adopt the weaker condition (21), and impose it as a boundary condition on the initial states.

The action including the fermions and the interaction is given by

$$S = \int d\tau \left( i \int d\sigma \left( \psi^\dagger \partial_\tau \psi + \int d\mathbf{z} \phi^\dagger \partial_\tau \phi \right) - H(\tau) \right), \quad (23)$$

with  $H$  given by eq.(18).

This action has an important advantage over the previous version given in references [3,4]: It is local in both the coordinates  $\sigma$  and  $\tau$ . We recall that the non-locality in [3] had two sources: The propagator must be attached to two adjacent solid lines; non-adjacent solid lines do not generate a propagator. This is because in the  $\mathbf{q}$  (potential) basis, the propagator is generated by a potential that satisfies boundary conditions on the adjacent solid lines. In [3], this condition was implemented by attaching a factor non-local in  $\sigma$  which vanishes for non-adjacent solid lines. This problem is avoided in the charge ( $\mathbf{z}$ ) basis: All charges, adjacent or non-adjacent, interact with the same one dimensional coulomb potential

$$\frac{1}{2} |\sigma - \sigma'|.$$

There is, however, one remaining problem: The prefactor  $1/(2p^+)$  in eq.(1) is missing. In [3, 4], this factor was attached to the interaction, and if we tried the same approach here, this would again have to introduce complicated non-local terms. Instead, as we will discuss in more detail later,  $1/(2p^+)$  is a measure factor needed in switching from the covariant propagator to the light cone propagator. In the manifestly Lorentz invariant version of the model, which will be the focus of our interest, there is no such extra factor, and this complication does not arise. Since we are using the above non-covariant model as an introduction to the covariant theory, we will leave it in its present simple though imperfect form.

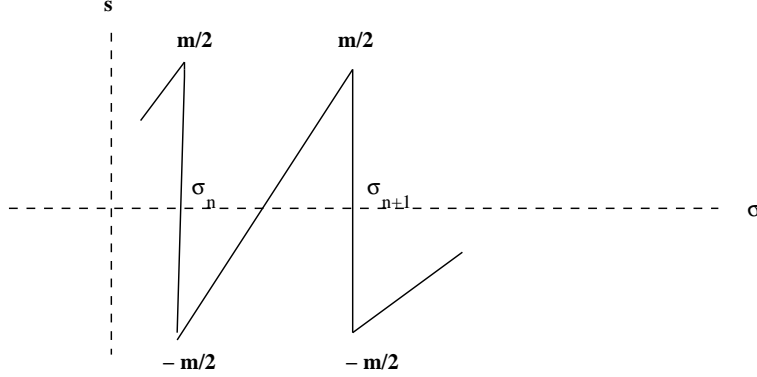


Figure 3: The Saw-tooth Configuration

## 6. The Mass Term

The model developed so far is massless. To introduce a finite mass term, we will add an extra space like dimension to the model. We could try compactifying the extra dimension, but this would generate a whole tower of Kaluza-Klein states. In order to generate a single massive particle, we again make use of the electrostatic analogy. Let the potential corresponding to the extra dimension be  $s(\sigma, \tau)$ , and let the solid lines be located at  $\sigma_n$ . In eq.(4), we make the replacements

$$\mathbf{q} \rightarrow s, \quad \mathbf{q}_{n+1} \rightarrow m/2, \quad \mathbf{q}_n \rightarrow -m/2,$$

and in the interval  $\sigma_n \leq \sigma \leq \sigma_{n+1}$ , we have

$$s(\sigma) = \frac{m}{2} \frac{2\sigma - \sigma_n - \sigma_{n+1}}{\sigma_{n+1} - \sigma_n}, \quad (24)$$

where  $m$  is the mass. The resulting saw-tooth configuration is sketched in Fig.3. It is easy to see that the corresponding action is again eq.(5), with, however, the replacement

$$(\mathbf{q}_{n+1} - \mathbf{q}_n)^2 \rightarrow m^2.$$

This is precisely the mass term in the propagator (1).

To construct the corresponding action, let us recall that in the case of the massless model, we placed charges on the solid lines, and the resulting potential reproduced the light cone propagator. The potential  $\mathbf{q}$  was continuous across a solid line, and the electric field jumped by minus the charge

**z.** In contrast,  $s$  is discontinuous at  $\sigma = \sigma_n$ , with a jump given by  $m$ . This corresponds, instead of charge, to a dipole of strength  $m$ . A simple action which takes the dipole sources into account is

$$S_m = \int d\tau \int d\sigma \left( -\frac{1}{2}(\partial_\sigma s)^2 - m s \partial_\sigma \rho \right), \quad (25)$$

leading to the equation of motion

$$\partial_\sigma^2 s = m \partial_\sigma \rho = m \sum_n \delta'(\sigma - \sigma_n), \quad (26)$$

which follows from eq.(22). This equation of motion admits the  $s$  given by (4) as a solution, and as we have already argued, this solution, substituted into the action reproduces the mass term in the propagator (1).

There is, however, still a remaining problem; although  $s$  given by (24) is a solution to (26), it is not the unique solution. The equation of motion for  $s$  only requires that crossing a solid line in the positive direction,  $s$  jumps by a total amount  $m$ , say from  $m_1$  to  $m_1 + m$ . It does not require it to jump from  $-m/2$  to  $+m/2$ ;  $m_1 = -m/2$  has to be imposed as an additional boundary condition. We have found that the simplest way to do this is by an orbifold projection. The action (25) is invariant under the  $Z_2$  symmetry

$$s(\sigma) \rightarrow -s(-\sigma), \quad \rho(\sigma) \rightarrow \rho(-\sigma), \quad (27)$$

assuming that the range of integration is symmetric under the reflection of  $\sigma$ . We now require the solutions to equations of motion (26) to be also invariant under this symmetry. From Fig.3, the saw-tooth configuration is clearly invariant under (27): Running  $\sigma$  backwards changes the sign of  $s$ . It is not difficult to show that the converse is also true: Symmetry under (27) fixes the solution to the equations of motion to be the saw-tooth; all other configurations are projected out.

## 7. Covariant World Sheet Field Theory For The Free Scalar

In this section, we will introduce a covariant generalization of the first quantized light cone world sheet action (eq.(2)), and show that it is correct by transforming it into its light cone version. This was first done in section 3 of [2], and so this section is mostly a review. We will, however, have some additions and amplifications; for example, the mass term (eq.(53)) is new.

We start with the the covariant action given in [2] :

$$S_0 = \int d\tau \int d\sigma \left( \frac{1}{2} \lambda \partial_\sigma q^\mu \partial_\sigma q_\mu + \sum_n \delta(\sigma - \sigma_n) y_n^\mu \partial_\tau q_\mu \right). \quad (28)$$

Comparing this with the light cone action (2), we notice the following:

a) As opposed to the transverse  $\mathbf{q}$ ,  $q^\mu$  and  $y_n^\mu$  are Minkowski vectors in  $D$  dimensions. As before,  $y_n$ 's are the Lagrange multipliers for the conservation equations

$$\partial_\tau q^\mu(\sigma_n, \tau) = 0. \quad (29)$$

b) The new field  $\lambda(\sigma, \tau)$  looks like a Lagrange multiplier. However, we shall see that it is closely related to the Schwinger proper time. It is therefore restricted to be positive semi-definite.

c) In the light cone action,  $\sigma$  and  $\tau$  are identified with  $p^+$  and  $x^+$  respectively. Here,  $\sigma$  and  $\tau$  are arbitrary parameters.

We will now proceed as at the beginning of section 3 by solving the equations of motion for  $q^\mu$  in terms of  $\tau$  independent vectors  $q_n^\mu = q^\mu(\sigma_n)$ . If  $\sigma$  lies in the interval  $\sigma_n \leq \sigma \leq \sigma_{n+1}$ , then

$$q^\mu(\sigma, \tau) = q_n^\mu + v_n^\mu(\tau) \int_{\sigma_n}^{\sigma} d\sigma' \lambda^{-1}(\sigma', \tau). \quad (30)$$

Setting  $\sigma = \sigma_{n+1}$  gives

$$v_n^\mu(\tau) = \frac{q_{n+1}^\mu - q_n^\mu}{\kappa_n(\tau)}, \quad (31)$$

where we have defined

$$\kappa_n(\tau) = \int_{\sigma_n}^{\sigma_{n+1}} d\sigma' \lambda^{-1}(\sigma', \tau). \quad (32)$$

Using these results. the action (28) becomes,

$$\begin{aligned} S_0 &= \int d\tau \sum_n \int_{\sigma_n}^{\sigma_{n+1}} d\sigma \left( \frac{1}{2} \lambda \partial_\sigma q^\mu \partial_\sigma q_\mu \right) \\ &= \int d\tau \sum_n \frac{(q_{n+1}^\mu - q_n^\mu)^2}{2\kappa_n(\tau)}. \end{aligned} \quad (33)$$

It is now easy to generalize to the massive case. Eq.(25) is replaced by

$$S_m = \int d\tau \int d\sigma \left( -\frac{\lambda}{2} (\partial_\sigma s)^2 - m s \sum_n \delta'(\sigma - \sigma_n) \right), \quad (34)$$

and with the addition of this term, the action becomes

$$S_0 = \int d\tau \sum_n \frac{(q_{n+1}^\mu - q_n^\mu)^2 - m^2}{2\kappa_n(\tau)} = \int d\tau \sum_n \frac{(p_n^\mu)^2 - m^2}{2\kappa_n(\tau)}. \quad (35)$$

We note that the final result for the action depends only on  $\kappa_n(\tau)$  and not on the full  $\lambda(\sigma, \tau)$ ; the  $\sigma$  dependence has been integrated over. This is a kind of gauge invariance: We can shift  $\lambda^{-1}$  by an arbitrary function of  $\sigma$  and  $\tau$ , so long as we keep  $\kappa_n$  fixed. By suitable gauge fixing, it is then possible to make  $\lambda$  independent of  $\sigma$  for each propagator separately.

Let us now focus on the contribution of a single interval in this action, say  $\sigma_n \leq \sigma \leq \sigma_{n+1}$ ; this term is invariant under the  $\tau$  reparametrization

$$q^\mu(\sigma, \tau) \rightarrow q^\mu(\sigma, f_n(\tau)), \quad \lambda(\sigma, \tau) \rightarrow f'_n(\tau) \lambda(\sigma, f_n(\tau)), \quad \kappa_n(\tau) \rightarrow \frac{\kappa_n(f_n(\tau))}{f'_n(\tau)}. \quad (36)$$

recalling that  $\lambda$  and  $\kappa$  are restricted to be positive, we also require

$$f'_n(\tau) \geq 0.$$

We will now assume that subject to these restrictions, it is possible to choose the  $f_n$ 's so as to map the  $\kappa_n$ 's into unity:

$$\kappa_n(\tau) \rightarrow \frac{\kappa_n(f_n(\tau))}{f'_n(\tau)} = 1. \quad (37)$$

If now the range of  $\tau$  in the  $n$ 'th term is from 0 to  $T_n$ , then, integrating over  $T_n$ ,

$$\int_0^\infty dT_n \exp \left( i \int_0^{T_n} ((p_n^2 - m^2)) \right) = \frac{i}{p_n^2 - m^2 - i\epsilon}, \quad (38)$$

we get the covariant Feynman propagator. Here  $T$  can be identified with the Schwinger proper time, after a Euclidean rotation.

The above derivation of the covariant propagator is somewhat heuristic; a more careful discussion of the  $\tau$  reparametrization fixing is needed. Rather than trying to clean up this derivation, we will instead review a better founded derivation of the light cone propagator given in [2]. As pointed out by Thorn [5], once the light cone propagator is established, one can switch to the Schwinger representation by means of the transformation

$$\begin{aligned} & \int \frac{dp^-}{2\pi} e^{-ix^+ p^-} \int dT \exp \left( -iT (\mathbf{p}^2 - 2p^+ p^- + m^2 - i\epsilon) \right) \\ &= \int dT \delta(x^+ - 2T p^+) e^{-T(\mathbf{p}^2 + m^2)}, \end{aligned} \quad (39)$$

where  $x^+$  is identified with the time  $\tau$ . The prefactor  $1/2p^+$  comes from changing the variable of integration from  $T$  to  $x^+ = \tau$ .

In the covariant approach given above, the  $y$ 's did not appear explicitly in the action; instead, the action was expressed in terms of the boundary values of the  $q$ 's which are  $\tau$  independent. We will now derive a new representation for the action, which is better suited to the light cone picture. In this representation, a mixed set of variables are used. First, transverse  $\mathbf{y}_n$ 's and  $y_n^-$ 's are integrated over, resulting in the conservation equations

$$\partial_\tau \mathbf{q}_n(\tau) = 0, \quad \partial_\tau q_n^+ = 0. \quad (40)$$

The action can now be written in terms of the above set of  $\tau$  independent variables, plus  $y_n^+(\tau)$  and  $q^-(\sigma, \tau)$ :

$$S_0 = \int d\tau \int d\sigma \left( -\frac{1}{2} \lambda (\partial_\sigma \mathbf{q})^2 + \lambda \partial_\sigma q^+ \partial_\sigma q^- - \sum_n \delta(\sigma - \sigma_n) q_n^- \partial_\tau y_n^+ \right). \quad (41)$$

We notice that this action is invariant under the following reparametrization:

$$q^\mu(\sigma, \tau) \rightarrow q^\mu(h(\sigma, \tau), \tau), \quad \lambda(\sigma, \tau) \rightarrow (\partial_\sigma h(\sigma, \tau))^{-1} \lambda(h(\sigma, \tau), \tau), \quad (42)$$

if  $h$  satisfies the conditions

$$h(\sigma_n, \tau) = \sigma_n,$$

so that the boundaries are kept straight and the last term in eq.(41) stays invariant.

We now make use of this invariance to cast (41) into the light cone action (2). First, transforming by means of a  $\tau$  independent  $h$ , we can set

$$\sigma_{n+1} - \sigma_n = q_{n+1}^+ - q_n^+ = p_n^+ \quad (43)$$

at some fixed  $\tau = \tau_0$ . Since both sides of this equation are  $\tau$  independent (see eq.(40)), it is then valid for all  $\tau$ . By fixing one  $\sigma$  to be zero at one boundary, we might as well set

$$\sigma_{n+1} = q_{n+1}^+ = p_n^+. \quad (44)$$

Next, transforming by a  $\tau$  dependent  $h$  that preserves the boundaries, we can set

$$\sigma = q^+(\sigma, \tau) \rightarrow \partial_\sigma q^+ = 1$$



in the bulk also, and not just on the boundaries. Substituting this in (41), the equation of motion for  $q^-$  in the bulk leads to

$$\partial_\sigma \lambda = 0, \quad (45)$$

so  $\lambda$  depends only on  $\tau$ , a result which we have already derived in a different way following eq.(35). This result only holds in the bulk; using the equations of motion with respect to  $q^-$  gives the jump of  $\lambda$  across the boundaries:

$$\lambda(\sigma_n + \epsilon, \tau) - \lambda(\sigma_n - \epsilon, \tau) = \partial_\tau y_n^+. \quad (46)$$

We now appeal the  $\tau$  reparametrization invariance (eq.(36)). By suitably choosing different  $f_n$ 's, we can completely eliminate the discontinuities in  $\lambda$ , and in view of (45),  $\lambda$  can be set to a constant over the whole world sheet. Without loss of generality, this constant can be taken to be one:

$$\lambda = 1. \quad (47)$$

From eq.(46), we also have,

$$\partial_\tau y_n^+ = 0. \quad (48)$$

At this point, we need to relate the  $y$ 's to the position coordinate  $x^\mu(\sigma, \tau)$ . It was shown in [2] that  $x^\mu(\sigma, \tau)$  is  $\sigma$  independent in the bulk, and its jump over the  $n$ 'th boundary is  $y_n^\mu$ . This is summarized by the equation

$$\partial_\sigma x^\mu(\sigma, \tau) = \sum_n \delta(\sigma - \sigma_n) y_n^\mu(\tau), \quad (49)$$

which can be taken as the definition of  $x^\mu$ .

Now, it follows from (48) and (49) that  $\partial_\tau x^+$  has no jumps on the boundaries, and therefore it is completely  $\sigma$  independent. Using an overall  $\tau$  reparametrization still at our disposal, we can therefore set

$$\partial_\tau x^+ = 1 \rightarrow x^+ = \tau. \quad (50)$$

To summarize, we have shown that, by suitable  $\sigma$  and  $\tau$  reparametrizations, one can impose the conditions

$$\lambda = 1, \quad \sigma = p^+, \quad \tau = x^+, \quad (51)$$

and these are all that is needed to map the covariant theory into the light cone model of the previous sections. We have not so far discussed the mass

term, but it is clear that  $\lambda = 1$  is sufficient to show the equivalence of the light cone and the covariant mass term. Of course, we still have to orbifold by the  $Z_2$  symmetry of (27).

Although the constraint  $\lambda = 1$  came at the end of a lengthy process of fixing the coordinates, it could have been derived more simply right at the beginning by appealing only to  $\tau$  reparametrization. We recall that the action does not depend on the full  $\lambda$  but only the  $\kappa_n(\tau)$ 's (eq.(35)). These can be set equal to unity by the  $\tau$  reparametrizations (eq.(37)). It is important to note that, by leaving the  $\sigma$  coordinate and the overall  $\tau$  reparametrization arbitrary, this can be done without imposing the conditions

$$\sigma = p^+, \quad \tau = x^+,$$

By suitable  $\tau$  reparametrization, we can therefore replace eq.(28) by

$$S_0 = \int d\tau \int d\sigma \left( \frac{1}{2} \partial_\sigma q^\mu \partial_\sigma q_\mu + \sum_n \delta(\sigma - \sigma_n) y_n^\mu \partial_\tau q_\mu \right), \quad (52)$$

and similarly, the mass term becomes

$$S_m = \int d\tau \int d\sigma \left( -\frac{1}{2} (\partial_\sigma s)^2 - m s \sum_n \delta'(\sigma - \sigma_n) \right). \quad (53)$$

These expressions for  $S_0$  and  $S_m$  will be the starting point for the covariant quantization in the next section. We note that they are manifestly covariant, and they can be obtained from the analogous light cone expressions by the simple substitution

$$\mathbf{q} \rightarrow q^\mu, \quad \mathbf{y} \rightarrow y^\mu. \quad (54)$$

For example, in analogy with eq.(6), we define the covariant charge by

$$z^\mu = -\partial_\tau y^\mu, \quad (55)$$

and note that this charge is also conserved :

$$\partial_\tau z^\mu = 0.$$

Eq.(7) also has its obvious covariant analogue.

The advantage of starting second quantization with eqs.(52) and (53) should now be clear: We can simply repeat the developments of sections 4, 5 and 6, with the obvious replacement of  $\mathbf{q}$  and  $\mathbf{z}$  by  $q^\mu$  and  $z^\mu$ . There is,

however, one remaining problem that needs to be clarified. By setting  $\lambda = 1$ , the equations of motion with respect to  $\lambda$  are lost. We already pointed out that these equations are too stringent; but we cannot just ignore them. In the next section, we will argue that they have to be replaced by weaker set of constraints.

Finally, a few comments on the  $1/(2p^+)$  factors. These are guaranteed to appear in the light cone picture by the requirement of Lorentz invariance. In fixing the  $\sigma$  coordinate by eq.(44), there is a possible measure factor (Jacobian) in the functional integral which we have neglected. We have not directly computed this factor; but indirectly, Lorentz invariance tells us that it should be proportional to  $1/p^+$ . In fact, we have already seen (eq.(39)) that transforming the covariant propagator into the light cone variables accounts for this factor.

## 8. Second Quantized Covariant World Sheet Field Theory

In the last section, we have seen that the action  $S_0 + S_m$ , generates a covariant first quantized free world sheet theory, which, upon fixing the coordinates, reduces to its light cone version. As we have pointed out earlier, it is difficult to introduce interaction in the first quantized theory, and so, following section 4, we second quantize the model. Actually, our job is easy; having worked out the non-covariant version in detail, all we have to do is guess the obvious covariant generalization. We start with eqs.(52) and (53) for the free first quantized action, and observe that, their second quantized version is the analogue of eq.(9):

$$\begin{aligned} S_0 &= \int d\tau \left( i \int d\sigma \int dz \phi^\dagger \partial_\tau \phi - H_0(\tau) \right), \\ H_0 &= \int d\sigma \left( -\frac{1}{2} (\partial_\sigma q^\mu)^2 - q^\mu \cdot \int dz \phi^\dagger z_\mu \phi \right), \end{aligned} \quad (56)$$

where the  $z$  integration over the full  $D$  dimensional space.  $S_m$  is the same as in eq.(25).

The field  $\phi$  now depends on  $z^\mu$  instead of  $\mathbf{z}$ , and satisfies the commutation relations

$$[\phi(\sigma, \tau, z), \phi^\dagger(\sigma', \tau, z')] = \delta(\sigma - \sigma') \delta^D(z - z'). \quad (57)$$

The definitions of the fermions and the  $\rho$ 's given in section 5 are unchanged. The definition of the allowed states (eq.(15)) goes over to

$$|s_a\rangle = \prod_n \left( \phi^\dagger(\sigma_n, z_n) \psi_2^\dagger(\sigma_n) \right) |0\rangle, \quad (58)$$

and the interaction Hamiltonian is the same as in eq.(17):

$$H_I = \int d\sigma \int dz g \left( \rho^+(\sigma) \phi(\sigma, z) + \rho^-(\sigma) \phi^\dagger(\sigma, z) \right). \quad (59)$$

The operator  $K$  (eq.(19)) is now defined by

$$K(\sigma, \tau) = \int dz \phi^\dagger(\sigma, \tau, z) \phi(\sigma, \tau, z) - \rho(\sigma, \tau). \quad (60)$$

As before, it commutes with the Hamiltonian and annihilates the allowed states:

$$K|s_a\rangle = 0. \quad (61)$$

Conversely, this can be taken as the definition of allowed states.

Finally, the covariant action is

$$S = \int d\tau \left( i \int d\sigma \left( \psi^\dagger \partial_\tau \psi + \int dz \phi^\dagger \partial_\tau \phi \right) - H(\tau) \right), \quad (62)$$

where,

$$H = H_0 + H_m + H_I.$$

$H_0$  and  $H_I$  are given above, and

$$H_m = \int d\sigma \left( \frac{1}{2} (\partial_\sigma s)^2 + m s \partial_\sigma \rho \right).$$

It is important to understand that the action is not the whole story: The constraints on the states are also crucial. They are restricted to be the allowed states (58), so they have to satisfy the constraint (61). In addition, the equations of motion for  $s$  in the mass term should be solved subject to the orbifold projection (27). The above covariant action, combined with the also covariant restrictions on the states, is the main result of this paper. In addition to being manifestly Lorentz invariant, it has the advantage of being local in the world sheet coordinates.

How do we know that this is the correct covariant field theory? To show this, we essentially follow the same steps as in sections 5 and 6. Using the interaction representation, we expand the hamiltonian in powers of the coupling constant  $g$ . Next, as in section 4, we consider the action of the free Hamiltonian on a typical allowed state. It is easy to show that the analogue of eq.(11),

$$q^\mu \cdot \int dz \phi^\dagger z_\mu \phi |s\rangle = \sum_n \delta(\sigma - \sigma_n) q_\mu z_n^\mu |s\rangle, \quad (63)$$

follows. Identifying  $z^\mu = -\partial_\tau y^\mu$ , this reproduces the covariant first quantized free action of eq.(52). The mass term is also easily seen to reproduce eq.(53). The interaction Hamiltonian, acting on states, either creates or annihilates a solid line (boundary) (Fig.2). The fermionic factor ensures that allowed states are mapped into other allowed states. All of this is in complete parallel with section 5. Once the covariant first quantized world sheet picture is established, we appeal to section 7 to show that it is equivalent to the light cone picture, which was the starting point. This completes the proof of equivalence between the covariant and the light cone theories. Once more, we stress that the restriction to the allowed states was crucial for the success of this proof of equivalence.

Finally, we would like to return to the constraint equation

$$L(\sigma, \tau) = (\partial_\sigma q^\mu)^2 - (\partial_\sigma s)^2 = 0, \quad (64)$$

obtained by varying with respect  $\lambda$ . It was pointed out in section 7 that, as an operator equation, this is too stringent: Going back to eqs.(4) and (24), in the interval  $\sigma_n \leq \sigma \leq \sigma_{n+1}$ ,

$$\partial_\sigma q^\mu = \frac{p_n^\mu}{\sigma_{n+1} - \sigma_n}, \quad \partial_\sigma s = \frac{m}{\sigma_{n+1} - \sigma_n},$$

and, therefore,

$$(\partial_\sigma q^\mu)^2 - (\partial_\sigma s)^2 \rightarrow \frac{(p_n^\mu)^2 - m^2}{(\sigma_{n+1} - \sigma_n)^2}. \quad (65)$$

This leads, for the n'th free propagator, to the mass shell condition

$$p_n^2 - m^2 = 0, \quad (66)$$

instead of the Feynman propagator. In any case, since  $\lambda$  is intrinsically positive, we cannot use it as a Lagrange multiplier. We therefore propose to replace (64) by the following weaker condition: Instead of vanishing as an operator,  $L$  should annihilate the allowed states:

$$L|s_a\rangle = 0, \quad (67)$$

This only puts the free particle states on the mass shell; it is therefore quite acceptable. However, we have so far checked the above constraint only in the context of a perturbation expansion.

It is quite possible that in a more general non-perturbative treatment, this has to be replaced by the even weaker condition: Only the positive frequency components of  $L$  may have to be required to annihilate the allowed states. Using mean field approximation, it was shown in [4,8] that a world sheet densely covered with graphs leads to string formation. This was done starting with the light cone picture, so the resulting string was a transverse light cone string. In contrast, if in some approximation, the covariant approach also leads to a string, that would be a covariant string, with its usual negative metric (ghost) modes. The above constraint may then be needed to eliminate these unphysical modes.

## 9. Conclusions And Future Directions

In this article, we have developed a covariant world sheet description of the planar  $\phi^3$  theory. Covariance is of course a desirable feature, and this new approach, in some ways, should be an improvement over the well developed light cone approach. It remains to be seen whether an approximation scheme, such as the mean field method, can be successfully adapted to this new model. Another important problem for future research is to try to generalize from  $\phi^3$  to more realistic theories, such as gauge theories <sup>4</sup>. It would very nice indeed to be able to formulate gauge theories in a manifestly both Lorentz and gauge invariant form.

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<sup>4</sup>For some initial steps taken towards more realistic theories, see [9, 10].

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